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Hamilton-Jacobi Analysis of Reparametrized Lagrangian Systems

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الآراء الواردة في الرسالة الجامعية لا تعبر بالضرورة عن وجهة نظر جامعة مؤتة

Dedication

To my mother's precious
To my father dear
To my brothers and sisters generous
Loyal to my teachers
To the care of warrants

Hassan Ruddah Althubyani

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Hassan Ruddah Althubyani

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ABSTRACT

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We have investigated equations of motion for systems that possesses the structure reparametrized invariant lagrangian systems with a particular interest in its applications. We applied the technique of separation of variables and the method of canonical transformations to solve the Hamilton-Jacobi equation. The quantization of reparametrized invariant lagrangian systems is investigated using the WKB approximation.

الملخص

تحليل هاميلتون_جاكوبي لنظم لاجرانج ذات بنية البارو مترية المعادة

حسن ردة الذبياني

جامعة مؤتة، 2012

قمنا باستقصاء معادلات الحركة لأنظمة لاجرانج التي تمتلك بنية البارومترية المعادة مع بعض الاهتمام في تطبيقاتها. طبقنا أسلوب فصل المتغيرات وطريقة الانتقالات الفيصلية لحل معادلات هاملتون جاكوبي. تكمية أنظمة لاجرانج البارومترية ثم اختبارها بطريقة الـ WKB .

Chapter One
Introduction

1.1 Statement of the Problem

There are two very useful methods in classical mechanics: the Hamiltonian and the Lagrangian approach (Doplicher *et al.*,1995; Bahns *et al.*,2002; Liao and Sibold,2002; Fujikawa,2004 & Balachandran *et al.*,2004). The Hamiltonian formalism gives rise to the canonical quantization, while the Lagrangian approach is used in the path-integral quantization. Usually, in classical mechanics, there is a transformation that relates these two approaches. However, for a reparametrization invariant systems there are problems when changing from the Lagrangian to the Hamiltonian approach (Muslih, 2002; Muslih,2002; Baleanu and Güler, 2003; Muslih, *et al.*,2004). Classical mechanics of a reparametrization invariant system and its quantization is the topic of the current study.

Parametrization invariance is a way that takes the time as an extra canonical variable of the system on the same footing as the position variable, and it is then easy to introduce a non-canonical structure in the extended phase-space by including an invariant parameter τ through the action integral which will play the role of the time. Hence, the canonical transformation here is implemented in an extended phase space, where the time and its conjugate momentum are included (Tkach *et al.*, 1999; Rabei *et al.*,2002, 2004; Nawafleh *et al.*,2004, 2005).

It is well known that any standard Hamiltonian system can be transformed to a constrained system with a vanishing Hamiltonian by going to an arbitrary reparametrization of time, thereby introducing the original time coordinate t as a new dynamical variable on the same footing as the position q.

Reparametrization theories of gravity such as general relativity and string theory are invariant under reparametrization of time (Dirac, 1967; Henneaux and Teitelboim,1992). A transformation from a reparametrization-invariant system to an ordinary gauge system was applied for deparametrizing cosmological models .

Reparametrization invariance was treated as a gauge symmetry in (Fülop *et al.*, 1941) and a time-dependent Schrödinger equation for systems invariant under the reparametrization of time was developed (Rosales *et al*, 2001; Tkach *et al.*,1999). To reach this goal an additional invariance action was introduced without changing the equation of motions but modifying the set of constraints.

The usual way to study the parametrization invariance of a system is by using the Dirac method of canonical analysis. Because not all the momenta are independent due to the invariance under parametrizations, this approach

requires that a constraint on the system be introduced. For a parametrized particle, this constraint is at the classical level the Hamilton-Jacobi equation and at the quantum level the Schrödinger equation. So the Dirac method associates to the symmetry of parametrizations the classical or quantum evolution equations (Henneaux and Teitelboim, 1992).

In Dirac's method, the constraints caused by the singularity of Hessian matrix are added to the canonical Hamiltonian, and then the consistency conditions are worked out, being possible to eliminate some degrees of freedom of the system (Dirac,1950 and Dirac,1964). Dirac also showed that the gauge freedom is caused by the presence of first class constraints. This formalism has a wide range of applications in field theory and it is still the main tool for the analysis of singular systems (Sundermeyer,1982; Gitman and Tyutin,1990). Despite the success of Dirac's method, it is always interesting to apply different formalisms to the analysis of singular systems.

Another powerful approach to study Hamilton-Jacobi (HJ) parametrization invariance of a system is by using the Hamilton-Jacobi formalism for constrained systems (Güler, 1992), based on Caratheodory's equivalent Lagrangians method (Caratheodory, 1967). This formalism does not differentiate between the first and second class constraints, we do not need any gauge fixing terms and the action provided by (HJ) is useful for the path integral quantization method of the constrained systems. In addition, it was proved that the integrability conditions of (HJ) formalism and Dirac's consistency conditions are equivalent (Pimentel *et al.*,1998) and the equivalence of the chain method (Rund, 1966; Mitra and Rajaraman, 1990) and (HJ) formalism was investigated (Baleanu and Güler, 2003).

In this thesis, we want to generalize the above mentioned procedure to write down the Hamilton-Jacobi equations for the system and make use of its singularity to write the equations of motion as total differential equations in many variables. The interesting point of the procedure is that on the one hand we get the classical and quantum evolution equations for the reparametrized invariant systems and on the other hand we also obtain a classical action that can be quantized using the WKB approximation (Rabei *et al.*,2002). Another interesting property of the method is that it can be naturally extended to field theory.

1.2 Motivation

Time plays a central and peculiar role in quantum mechanics. In the standard nonrelativistic quantum mechanics, one can describe the motion of a system by using the canonical variables which are functions of time only. Time is the sole observable assumed to have a direct physical significance; but it is not a dynamical variable itself. It is an absolute

parameter treated differently from the other coordinates, which turn out to be operators in quantum mechanics.

In the cases of nonrelativistic and relativistic point-particle mechanics, covariant systems may generally be obtained by promoting the time to a dynamical variable. The idea behind this transformation is to treat the time and the dynamical variables of the system symmetrically. This is achieved by taking t as a function of an arbitrary parameter τ: $t=t(\tau)$, $q=q(\tau)$ (e.g., τ is the "proper time" in relativity theory). The arbitrariness of the label time τ is reflected in the invariance of the action under the time reparametrization. Thus, we can express the action integral with respect to τ in the same form as with respect to t. This shows that the equations of motion which follow from the action principle must be invariant under the transformation from t to τ.

1.3 Synopsis of the thesis

The plan of this thesis is as follows: in chapter one, a brief introduction to the reparamerized invariant Lagrangian systems is presented. In chapter two the theoretical framework is introduced in three sections: section one, contains Lagrangian mechanics and Hamiltonian mechanics. The second one presents Hamilton-Jacobi treatment of of reparametrized systems. The third section includes the WKB approximation of reparametrized Lagrangian.

Chapter three contains illustrative examples comes in two sections, the first one represents the motion of nonrelativistic particle system, and the second section includes the motion of relativistic particle system.

Chapter four is divide to two sections, the first one contains the classical field systems, and the second one includes the motion of a relativistic spineless particle. Finally, chapter five devoted to final conclusions.

Chapter Two
Theoretical Framework

This chapter is concerned with the theoretical framework for Hamilton-Jacobi formalism of reparametrized systems. In Section (2.1) we review theory for lagrangian mechanics and Hamilton mechanics. In Section (2.2) we introduce Hamilton-Jacobi formalism of reparametrized systems. Finally we introduce the WKB approximate for reparametrized Lagrangian systems in section (2.3).

2.1 Review Of Lagrangian Mechanics and Hamiltonian Mechanics
2.1.1 Lagrange's Equation

The formulation of Lagrange's equations begins with the concept of a functional, $(q,\dot{q})$ any function of some class of curves (Landau and Lifshitz, 1976), we consider a system that occupies positions $q(t_0)$ and $q(t_1)$ at times t_0 and t_1, respectively. The fundamental functional in the Lagrangian formalism is the action

$$S = \int_{t_0}^{t_1} L(q,\dot{q})\,dt \,. \tag{2.1}$$

The function L is called the lagrangian of the system.

The requirement that S be minimized implies that the variation of S vanishes,

$$\delta S = \delta \int_{t_0}^{t_1} L(q,\dot{q})dt = 0 .$$

This condition leads to

$$\delta S = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right) dt = 0 .$$

Integrating the second term by parts, we obtain

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}}\delta q \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q\, dt = 0 . \tag{2.2}$$

At the endpoints the variation of q is zero, that is, $\delta q(t_0) = \delta q(t_1) = 0$.
Therefore, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \tag{2.3}$$

Which is known as the Euler–Lagrange equation (Goldstein, (1980); Hand and Finch, 2008).

2.1.2 Hamilton's Equations

The Hamilton's function is the Legender transform of the Lagrangian with respect to the variable $\dot{q}$ (Landau and Lifshitz, 1976; Goldstein, 1980)

$$H(p,q) = p\dot{q} - L(q,\dot{q},t) \qquad (2.4)$$

The total differential of the Hamiltonian is

$$dH = p_i d\dot{q}^i + \dot{q}^i dp_i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i - \frac{\partial L}{\partial t} dt \qquad (2.5)$$

Using eq.(2.3),we obtain

$$dH = -\dot{p}_i dq^i + \dot{q}^i dp_i - \frac{\partial L}{\partial t} dt. \qquad (2.6)$$

On the other hand, the Hamiltonian is a function of q and p, therefore its total differential has the form

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i\right) + \frac{\partial H}{\partial t} dt \qquad (2.7)$$

Comparing this with (2.6), the Hamilton equations are obtained as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \qquad (2.8)$$

2.1.3 Hamilton–Jacobi equation

As we derived in eq. (2.2), the variation of the action is

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}} \delta q\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right) \delta q\, dt = 0$$

The second term vanishes because the motion satisfies Lagrange's equations. In the first term, we set $\delta q(t_0) = 0$ and replace $\delta q(t_1)$ with δq because t_1 adopts any value of t greater than t_0. Using the relationship (2.3) we arrive at the equivalence $\delta S = p\delta q$. For an n dimensional system, this has the form

$$\delta S = p_i \delta q^i \qquad i=1,2,\ldots,n \qquad (2.9)$$

A relationship between the action and momenta follows directly,

$$p_i = \frac{\partial S}{\partial q_i} \qquad (2.10)$$

Another necessary result is produced by examining the total time derivative of the action. Directly from the definition of the action (2.1), we observe that

$$\frac{dS}{dt} = L \qquad (2.11)$$

However, by viewing the action as a function of only coordinates and time, it is obvious, using eq. (2.10), that

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q_i}\dot{q}_i = \frac{\partial S}{\partial t} + p_i\,\dot{q}_i \tag{2.12}$$

Comparing eq. (2.11) and eq. (2.12) , then using eq. (2.4), we obtain

$$\frac{\partial S}{\partial t} + H(p,q,t) = 0 \tag{2.13}$$

The momenta in eq. (2.13) can be replaced using eq. (2.10) to produce the first order partial differential equation called the Hamilton-Jacobi equation , (Benenti, 1989; Benenti, 1992)

$$\frac{\partial S}{\partial t} + H(q_1,....,q_n,\frac{\partial S}{\partial q_1},....,\frac{\partial S}{\partial q_n},t) = 0 \tag{2.14}$$

We now establish the connection between the complete integral of the Hamilton-Jacobi equation and solutions of Hamilton's equations. We use the function W(q,t;α) as the generating function for a canonical transformation from the original coordinates (p,q) to (α,β); therefore, our new position coordinates are $(\beta_1,\beta_2,\ldots,\beta_n)$ and new momenta coordinates are $(\alpha_1,\alpha_2,\ldots,\alpha_n)$. with this generating function, the new Hamiltonian vanishes everywhere; thus, the transformed Hamilton's equations become (Ashraf, 2010)

$$\dot{\alpha}_i = 0\;,\;\dot{\beta}^j = 0$$

And we solve for the position coordinates q as functions of t, α and β using the relationships

$$\beta^i = \frac{\partial W}{\partial \alpha_i}$$

An important technique for the determination of complete integral for the Hamilton-Jacobi equation (HJE) of the system is the method of separation of variables. Under certain conditions it is possible to separate the variables in the HJE, the solution can be then always reduced to quadrature (Arnold, 1989; Brack and Bahaduri, 1997). In practice, the Hamilton-Jacobi technique becomes a useful computational tool only when such a separation can be effected. HJE play a good role and becomes beautiful treatment when it can be solved using separation of variables, which directly identifies constant of motion.

In general, a coordinate q_i is said to be separable in the HJ equation when Hamilton's principal function S can be split into two additive parts, one of which depends only on the coordinate q_i ; whereas the second is independent of q_i , which means time dependent part.

In the cases to which we shall apply the method of separation of variables, the Hamiltonian will be time independent. Therefore the HJE (2.14) for this system will be in the following form:

$$\frac{\partial S(q,t,\alpha)}{\partial t} + H\ (q, \frac{\partial S}{\partial q}) = 0$$

We can separate the variable as

$$S(q,\alpha,t) = W(q,\alpha) + f(t) + A\ , \tag{2.15}$$

where the time- independent function W(q,α) is sometimes called Hamilton's characteristic function.

Differentiate Eq (2.15) with respect to time, we find

$$\frac{\partial S}{\partial t} = \frac{\partial f}{\partial t},$$

and using the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\ = 0, \tag{2.16}$$

then, we have

$$\frac{\partial f}{\partial t} = -H = -\alpha \tag{2.17}$$

The left hand sides of equation (2.17) depends on t, whereas the right hand side depends on q, so that each side equal to a constant independent of both q and t, the time t can be separated if the Hamiltonian does not depend on time explicitly. In that case, the time derivative $\frac{\partial\ S}{\partial\ t}$ in the HJE must be a constant, usually denoted by (- α), giving the separated solution. Then we obtain:

$$\mathrm{S}(\mathrm{q},\alpha,\mathrm{t}) = \mathrm{W}(\mathrm{q},\alpha) - \alpha\,\mathrm{t} + \mathrm{A}\,. \tag{2.18}$$

2.2 Hamilton-Jacobi Treatment of Reparametrized Systems

In the cases of no relativistic and relativistic point-particle mechanics, generally covariant systems may be obtained by promoting *t* to a dynamical variable. The idea behind this transformation is to treat symmetrically the time and the dynamical variables of the system. This is achieved by taking *t* as a function of an arbitrary parameter τ : $t=t(\tau), q=q(\tau)$ (e.g., τ is the "proper time" in relativity theory). The arbitrariness of the label time τ is reflected in the invariance of the action under the time reparametrization. If S is the action integral, then

$$S = \int L(q, \frac{dq}{dt})dt = \int L^*(q, \frac{dq}{d\tau})\, d\tau\,. \tag{2.19}$$

Thus, we can express the action integral with respect to τ in the same form as with respect to t. This shows that the equations of motion which follow from the action principle must be invariant under the transformation from t to τ. The equations of motion do not refer to any absolute time. We

have, therefore, a special form of Hamiltonian theory; but this form is not really so special because, starting with any Hamiltonian, it is always permissible to take the time variable as an extra coordinate and bring the theory into a form in which the Hamiltonian is equal to zero (Dirac, 1967; Rosales, 2001; De Cicco and Simeone,1999). The general rule for doing this is the following: we take t and put it equal to another dynamical coordinate q_0. We set up a new Lagrangian:

$$L^* = L^*\left(q_0, q_i, \frac{dq_0}{d\tau}, \frac{dq_i}{d\tau}\right), \qquad i = 1, 2, \dots, N \qquad (2.20)$$

L^* involves one more degree of freedom than the original L.

L^* is not equal to L; but

$$S = \int L\left(q_i, \frac{dq_i}{dq_t}\right) dt = S^* = \int L^*\left(q_0, q_i, \frac{dq_0}{d\tau}, \frac{dq_i}{d\tau}\right) d\tau \qquad (2.21)$$

Thus, the action is the same whether it refers to L^* and τ or to L and t.

This special case of the Hamiltonian formalism, where the Hamiltonian

$$H_0^* = p_0 \frac{dq_0}{d\tau} + p_i \frac{dq_i}{d\tau} - L^* \equiv 0, \qquad (2.22)$$

is what is needed for a relativistic theory, because in such a theory we do not want to have one particular time playing a special role. Instead, we want to have the possibility of various times τ which are all on the same footing.

However, for reparametrized systems, the Hamilton-Jacobi equation takes the form (Guler,1992)

$$H_t' = p_t + H_t = 0 \qquad (2.23)$$

p_t being the generalized momentum associated with t and H_t of the originally no covariant formulation.

2.3 WKB Approximation of Reparametrized Lagrangians

It is well known that the HJ equation for dynamical systems leads naturally to a semiclassical approximation; namely, WKB. The Schrödinger equation in one dimension for a single particle in a potential *V* (*q*) reads

$$i\hbar \frac{\partial \psi(q,t)}{\partial t} = \left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} + V(q)\right]\psi(q,t) \qquad (2.24)$$

Writing $\psi(q,t) = \exp\left(\frac{iS(q,t)}{\hbar}\right)$ and considering the expansion

$$S(q,t) = S_0 + \hbar S_1 + \hbar^2 S_2 + \dots..$$

such an expansion is used in the WKB approximation where S is real to leading order; at this point we have not said anything about the reality of S, so the above equation is just a mathematical identity. Then we have

$$-\frac{\partial S}{\partial t}=-\frac{i\hbar}{2m}\frac{\partial^2 S}{\partial q^2}+(\frac{\partial S}{\partial q})^2+V(q) \qquad (2.25)$$

If we assume that $\hbar \to 0$, which is the 'classical limit' in quantum mechanics, then we see that

$$-\frac{\partial S}{\partial t}=(\frac{\partial S}{\partial q})^2+V(q)$$

More general as

$$\frac{\partial S}{\partial t}+H(q,\frac{\partial S}{\partial q})=0 \qquad (2.26)$$

which is just the Hamilton-Jacobi equation. Thus we see that in the classical limit $\hbar \to 0$ the Schrödinger equation is just the Hamilton-Jacobi equation when the dynamical coordinates and moment are turned into their corresponding operators :

$$q_i \to q_i \qquad (2.27)$$

$$p_i \to \hat{p}_i=\frac{\hbar}{i}\frac{\partial}{\partial q_i}$$

$$p_o \to \hat{p}_o=\frac{\hbar}{i}\frac{\partial}{\partial t}$$

In the classical limit $\hbar \to 0$, the condition (2.23) implies that

$$\hat{H}'_t\psi=o \qquad (2.28)$$

Chapter Three
Illustrative Examples

3.1 Nonrelativistic Reparametrized Particle System

In this section the central idea is illustrated with the aid of a simple model of reparametrized dynamics. We start by considering the theory of a no relativistic particle moving in one-dimensional space with dynamical variables x and with t denoting the ordinary physical time parameter. The action for this simplest model may be written as

$$S = \int L dt = \int \left(\frac{1}{2} m \dot{x}^2 - V(x) \right) dt \ , \tag{3.1}$$

Where m is the mass of the particle, $\dot{x} = dx/dt$ is its velocity and $V(x)$ is the potential.

In the action (3.1) time t is an absolute parameter, taking t to be a function of local time τ, t = t(τ), then eq. (3.1) reduces to

$$S^* = \int L^* d\tau = \int \left(\frac{1}{2} m \frac{\dot{x}^2}{\dot{t}} - V(x)\dot{t} \right) d\tau$$

where

$$L^* = L\dot{t} = \frac{1}{2} m \frac{\dot{x}^2}{\dot{t}} - V(x)\dot{t} \tag{3.2}$$

the dot now standing for $\frac{d}{d\tau}$.

The generalized momenta corresponding to L^* are

$$p_x = \frac{\partial L^*}{\partial \dot{x}} = \frac{m\dot{x}}{\dot{t}} \tag{3.3}$$

$$p_t = \frac{\partial L^*}{\partial \dot{t}} = -\frac{1}{2} m \frac{\dot{x}^2}{\dot{t}^2} - V(x) \equiv -H_t \tag{3.4}$$

Substituting the value of $\dot{x}$ from eq. (3.3) into eq. (3.4), then we have

$$p_t = -\frac{p_x^2}{2m} - V(x) \equiv -H_t$$

This equation can be written as:

$$H_t' = p_t + H_t = 0 \ , \tag{3.5}$$

where

$$H_t = \frac{p_x^2}{2m} + V(x)$$

It is easy to show that the canonical Hamiltonian H_0^* is identically zero

$$H_0^* = p_t \dot{t} + p_x \dot{x} - L^* = 0$$

The corresponding Hamilton-Jacobe Partial differential equation HJPDE of eq. (3.5), is

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2+V(x)=0 \qquad (3.6)$$

Using the separation of variables technique we can write

$$S(x,\alpha,t)=-\alpha t+W(\alpha,x)+A\,. \qquad (3.7)$$

Then eq. (3.6) becomes

$$-\alpha+\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2+V(x)=0\,,$$

which can be solved for W(x,α) as

$$W(x,\alpha)=\int\sqrt{2m(\alpha-V)}dx\,.$$

Therefore the general solution of S takes the form

$$S(x,\alpha,t)=-\alpha t+\int\sqrt{2m(\alpha-V)}dx+A\,, \qquad (3.8)$$

where A is a constant.
The equations of motion for x and p_x are

$$\beta=\frac{\partial S}{\partial\alpha}=-t+\int\frac{m}{\sqrt{2m(\alpha-V)}}dx\,, \qquad (3.9)$$

$$p_x=\frac{\partial S}{\partial x}=\sqrt{2m(\alpha-V)}\,. \qquad (3.10)$$

Equations (3.9) and (3.10) can be sloved for x and p_x in terms of t and α.
Applying now the operator H'_t, eq.(3.5), to the wave equation $\psi(x,t)$

$$H'_t\psi=\left[\frac{\hbar}{i}\frac{\partial}{\partial t}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\psi\,, \qquad (3.11)$$

With $\Psi(x,t)=\exp(\frac{iS}{\hbar})$, and S is given in eq.(3.8), then we have

$$H'_t\psi=\left[-\alpha-\frac{\hbar^2}{2m}\left(\frac{-2m}{\hbar^2}(\alpha-V)-\frac{im}{\hbar\sqrt{2m(\alpha-V)}}\frac{\partial V}{\partial x}\right)+V\right]\psi\,.$$

After simplifications we obtain

$$H'_t\psi=\left[-\alpha+\alpha-V+\frac{i\hbar}{2\sqrt{2m(\alpha-V)}}\frac{\partial V}{\partial x}+V\right]\psi \qquad (3.12)$$

In the semiclassical limit $\hbar\to 0$, eq.(3.12) is identically zero, that is

$$H'_t\Psi\equiv 0 \qquad (3.13)$$

3.2 Relativistic Reparametrized Particle System

Using the physical coordinates x(t), the relativistic particle action reads

$$S=\int Ldt=-mc\int\sqrt{c^2-\left(\frac{dx}{dt}\right)^2}dt\,, \qquad (3.14)$$

where m is the mass of the particle, and c is the speed of the light. Introducing an arbitrary paramertrization $x(\tau)$, $t(\tau)$ of the trajectory, the action requires the reparametrization invariant form

$$S^* = \int L^* d\tau = -mc \int \dot{t} \sqrt{c^2 - \frac{\dot{x}^2}{\dot{t}^2}} d\tau \,, \qquad (3.15)$$

or

$$S^* = \int L^* d\tau = -mc \int \sqrt{c^2 \dot{t}^2 - \dot{x}^2} d\tau \,, \; (3.16)$$

where

$$L^* = -mc \sqrt{c^2 \dot{t}^2 - \dot{x}^2} \qquad (3.17)$$

We introduce the generalized canonical momenta as

$$p_x = \frac{\partial L^*}{\partial \dot{x}} = \frac{mc\dot{x}}{\sqrt{c^2 \dot{t}^2 - \dot{x}^2}} \,, \qquad (3.18)$$

$$p_t = \frac{\partial L^*}{\partial \dot{t}} = \frac{-mc^3 \dot{t}}{\sqrt{c^2 \dot{t}^2 - \dot{x}^2}} \equiv -H_t \,. \qquad (3.19)$$

In fact eq. (3.19) may be written as

$$H_t' = p_t + H_t = 0 \,, \qquad (3.20)$$

and eq. (3.18) can be solved for $\dot{x}$

$$\dot{x} = \frac{c\dot{t} p_x}{\sqrt{p_x^2 + m^2 c^2}} \,. \qquad (3.21)$$

Then eq. (3.20) reduces to

$$H_t' = p_t + c\sqrt{p_x^2 + m^2 c^2} = 0 \,, \qquad (3.22)$$

It is obvious to note that H_0^* is identically zero:

$$H_0^* = p_t \dot{t} + p_x \dot{x} - L^* \equiv 0 \,. \qquad (3.23)$$

In details we have

$$H_o^* = \frac{-mc(c^2 \dot{t}^2 - \dot{x}^2)}{\sqrt{c^2 \dot{t}^2 - \dot{x}^2}} + \frac{mc(c^2 \dot{t}^2 - \dot{x}^2)}{\sqrt{c^2 \dot{t}^2 - \dot{x}^2}} = 0 \,. \qquad (3.24)$$

The corresponding HJPDE for eq.(3.22) is

$$\frac{\partial S}{\partial t} + c\sqrt{\left(\frac{\partial S}{\partial x}\right)^2 + m^2 c^2} = 0 \qquad (3.25)$$

Making use the separation of variables techniques we can write

$$S(x,\alpha,t) = -\alpha t + W(\alpha,x) + A \,, \qquad (3.26)$$

where A is a constant.

Therefore, Eq.(3.26) can be solved for W(x,α) as

$$W(x,\alpha) = \int \sqrt{\left(\frac{\alpha^2}{c^2} - m^2 c^2\right)} dx \qquad (3.27)$$

It follows that

$$S = -\alpha t + \int \sqrt{\left(\frac{\alpha^2}{c^2} - m^2 c^2\right)} dx + A. \qquad (3.28)$$

Now the equations of motion for x and p_x can be obtained from

$$\beta = \frac{\partial S}{\partial \alpha} = -t + \int \frac{\alpha}{c^2 \sqrt{\frac{\alpha^2}{c^2} - m^2 c^2}} dx, \qquad (3.29)$$

and

$$p_x = \frac{\partial S}{\partial x} = \sqrt{\left(\frac{\alpha^2}{c^2} - m^2 c^2\right)}. \qquad (3.30)$$

Following the previous procedure for the quantization of the system. With

$$\psi(x,t) = \exp\left(\frac{i}{\hbar}\left[-\alpha t + \int \sqrt{\left(\frac{\alpha^2}{c^2} - m^2 c^2\right)} dx + A\right]\right) \qquad (3.31)$$

The result of the operation $H_t'\psi$ reads

$$H_t'\psi(x,t) = \left[\frac{\hbar}{i}\frac{\partial}{\partial t} + c\sqrt{-\hbar^2 \frac{\partial^2}{\partial x^2} + m^2 c^2}\right]\psi(x,t)$$

Then we have

$$H_t'\psi(x,t) = \left[-\alpha + c\sqrt{-\hbar^2 \frac{\partial^2}{\partial x^2} + m^2 c^2}\right]\psi(x,t) \qquad (3.32)$$

In the semiclassical limit $\hbar \to 0$, this implies that

$$H_t'\psi(x,t) \equiv 0 \qquad (3.33)$$

Chapter Four
Reparametrization Invariant Fields

4.1 Classical Field Systems

The dynamics of continuous systems is described by function $Q(x)$ of space-time rather than the functions of time $q_i(t)$ in a discrete system. The discrete label i is replaced by the continuous label $x \equiv (ct, x)$ (Goldstein, 1980). Furthermore, in continuous systems, the function of the coordinates $f(q)$ becomes the function $F(Q)$ of fields.

The most general form of the Lagrangian in the field theory is functional of fields as well as their time and space derivatives, that is $x \equiv (ct, x)$.

$$L = \int \ell \, d^3x \tag{4.1}$$

where

$$\ell = \ell(Q_r, \partial^\mu Q_r), \qquad r = 1,2,3; \qquad \mu = 0,1,2,3 \tag{4.2}$$

is the corresponding Lagrangian density with

$$\partial^\mu Q_r \equiv \frac{\partial Q_r}{\partial x_\mu} \tag{4.3}$$

At this point we must decide on a metric convention for treating covariant vectors in four-dimensional space-time. The relation between the covariant vector A_μ and its contravariant partner A^μ is defined as $x \equiv (ct, x)$ (Sokolnikoff,1964; Jackson,1974) :

$$A_\mu = g_{\mu\nu} A^\nu \qquad \mu\nu = 0,1,2,3 \tag{4.4}$$

whereas its inverse is defined as:

$$A^\mu = g^{\mu\nu} A_\nu \tag{4.5}$$

here $g_{\mu\nu} = g_{\nu\mu}$ is called the metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.6}$$

For flat space-time of special relativity (in distinction to the curved space-time of general relativity), $g_{\mu\nu}$ is the same as $g^{\mu\nu}$, that is:

$$g_{\mu\nu} = g^{\mu\nu} \tag{4.7}$$

4.2 The Relativistic Spineless Particle System

Let us consider the action of the free relativistic spinless particle moving in 4 dimensional Minkowski space (4-component Lorentz

vector) x_μ (Gitman and Tyutin,1990; Sundermeyer,1982; Muslih and Güler,1998)

$$S = \int \left(-mc\sqrt{\dot{x}_\alpha \dot{x}^\alpha} - \frac{e}{c}\dot{x}_\alpha A^\alpha \right) d\tau \,, \; \alpha = 0,1,2,3 \qquad (4.8)$$

Here

$$L^* = -mc\sqrt{\dot{x}_\alpha \dot{x}^\alpha} - \frac{e}{c}\dot{x}_\alpha A^\alpha \,.$$

The generalized momenta p_α conjugated to the coordinates x_α are

$$p^\alpha = \frac{\partial L^*}{\partial \dot{x}^\alpha} = -mc\frac{\dot{x}_\alpha}{\sqrt{\dot{x}_\alpha \dot{x}^\alpha}} - \frac{e}{c}A_\alpha \,.$$

Therefore the zeroth component is

$$p^0 = -mc\frac{\dot{x}_0}{\sqrt{\dot{x}_\alpha \dot{x}^\alpha}} - \frac{e}{c}A_0 \qquad (4.9)$$

and the a^{th} components are

$$p^a = mc\frac{\dot{x}^a}{\sqrt{\dot{x}_\alpha \dot{x}^\alpha}} + \frac{e}{c}A^a, \qquad a = 1,2,3. \qquad (4.10)$$

Making use of eq.(4.10), one can solve for $\dot{x}^a$ in terms of the generalized momenta, $\vec{p}$, as

$$\dot{x}^a = \frac{\dot{x}_0\left(p^a - \frac{e}{c}A^a\right)}{\sqrt{m^2c^2 + \left|\vec{p} - \frac{e}{c}\vec{A}\right|^2}} \,.$$

Substitute the value of $\dot{x}^a$ into Eq.(4.9), then we have

$$p_0 = -\sqrt{m^2c^2 + \left|\vec{p} - \frac{e}{c}\vec{A}\right|^2} - \frac{e}{c}A_0 \equiv -H_0,$$

This equation can be written as

$$H_0' = p_0 + H_0 = 0 \qquad (4.11)$$

where

$$H_0 = \sqrt{m^2c^2 + \left|\vec{p} - \frac{e}{c}\vec{A}\right|^2} + \frac{e}{c}A_0 \,. \qquad (4.12)$$

One can show that the usual Hamiltonian $H_0{}^*$ vanishes identically, that is

$$H_0{}^* = p_0\dot{x}_0 + p_a\dot{x}_a - L^* \equiv 0 \,.$$

This result, $H_0^* = 0$, reflects the fact that the system described by L^* does not have a unique time parameter. The system is time-reparametrization invariant.

For more simplification, let us consider the motion of a spinless particle initially at rest and starts from the origin in a uniform electric field, E_0, directed along the z-axis: $\vec{E} = E_0\hat{z}$. In this case, the components of A^α can be obtained as

$$A^0 = \phi = -\int E_0 dz = -E_0 z \, .$$

and

$$A^a = 0 \, .$$

Thus, Eq.(4.11) becomes

$$H_0' = p_0 + \sqrt{p_z^{\,2} + m^2c^2} - \frac{eE_0 z}{c} = 0, \tag{4.13}$$

and the corresponding HJPDE is

$$\frac{\partial S}{\partial t} + \sqrt{\left(\frac{\partial S}{\partial z}\right)^2 + m^2c^2} - \frac{eE_0 z}{c} = 0 \, .$$

The general solution for this equation can be obtained as

$$S(z, \alpha, t) = f(t) + W(z, \alpha) + A \, .$$

With $f(t)=-\alpha t$, we find

$$W(z,\alpha) = \int \sqrt{\left(\frac{eE_0 z}{c} + \alpha\right)^2 - m^2c^2} \, dz \, .$$

It follows that

$$S = -\alpha t + \int \sqrt{\left(\frac{eE_0 z}{c} + \alpha\right)^2 - m^2c^2} \, dz \, .$$

Making use the canonical transformations, we find

$$\beta = \frac{\partial S}{\partial \alpha} = -t + \int \frac{[eE_0 z/c + \alpha]}{\sqrt{[eE_0 z/c + \alpha]^2 - m^2c^2}} dz, \tag{4.14}$$

$$p_z = \frac{\partial S}{\partial z} = \sqrt{\left(\frac{eE_0 z}{c} + \alpha\right)^2 - m^2c^2} \, . \tag{4.15}$$

Equation (4.14) can be integrated to give

$$z = \frac{c}{eE_0}\sqrt{[eE_0(\beta + t)]^2 + m^2c^2} - \frac{\alpha}{eE_0},$$

and the initial condition implies that

$$z = \frac{c}{eE_0}\sqrt{(eE_0 t)^2 + m^2c^2} - \frac{mc^2}{eE_0} \, . \tag{4.16}$$

Substituting for z in eq.(4.15), we get

$$p_z = eE_0 t\ .$$

That is the motion is linearly in time in the z-direction.

At the quantum level the effect of the operator H_0' on the wave function ψ in the semiclassical limit $\hbar \to 0$, implies that

$$\hat{H}_0'\psi \equiv 0$$

where

$$\hat{H}_0' = \left[\frac{\hbar}{i}\frac{\partial}{\partial t} + \sqrt{-\hbar^2 \frac{\partial^2}{\partial z^2} + m^2 c^2} - \frac{eE_0 z}{c} \right],$$

and

$$\psi = \exp\left[\frac{i}{\hbar}\left(-\alpha t + \int \sqrt{\left(\frac{eE_0 z + E}{c} \right)^2 - m^2 c^2}\ dz \right) \right].$$

Chapter Five
Conclusion

The Hamilton-Jcobi partial differential equations for reparametrized Lagrangian systems are discussed using the canonical method. It has been shown that any standard Hamiltonian system can be transformed into a constrained system with vanishing Hamiltonian by going to an arbitrary reparametrization of time. In doing so, the time variable is treated on the same level as the other dynamical variables. Thus, we have an extended phase space that includes a new coordinate, the time, whose conjugate momentum represents the total energy of the system.

Due to the reparametrization invariance, the quantity H_t' vanishes for any solution, $H_t' = p_t + H_t = 0$. So the corresponding quantum-mechanical operator annihilates the wave function $H_t'\psi = 0$, which is precisely the Schrödinger equation, $i\hbar\frac{\partial\psi}{\partial t} = H_t'\psi$.

Further, the Hamilton-Jacobi function S is determined in configuration space in the same manner as for regular systems. Finding S enables us to get the solutions of the equations of motion. These solutions are obtained in terms of the time and the coordinates that correspond to dependent momenta.

The success of this work has been demonstrated for three applications. The first is an illustrative example in one-dimensional dynamics that describes the concept of nonrelativistic parametrized dynamics. It has been shown that the quantization procedure applied to the initial mechanical system, after promoting the time to become a dynamical variable, yields the correct equation for the wave function, which is just the conventional time-dependent Schrödinger equation. The second application is quantization of the motion of a relativistic parametrized particle system. The third application is the motion of a spinning point-particle in an external electromagnetic field.

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